On the 3-cycle of the logistic map

Cheng Zhang

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We review a simple derivation of the onset and bifurcation points the 3-cycle of the logistic map.

The logistic map is defined as

$$z_{n+1} = F(z_n) = r z_n (1 - z_n).$$

We want to find the r values that permit a *stable* 3-cycle, i.e., a solution of $z_{n+3} = z_n$, which can sustain a small deviation in the initial z_0. The two boundaries r_0 and r_1 of the stable window (r_0, r_1) are called the *onset* and *bifurcation* points, respectively. By a change of variables

$$x_n = r(z_n - 1/2), R = (r^2 - 2r)/4, \tag{1}$$

we get a simplified map

$$x_{n+1} = f(x_n) = R - {x_n}^2.$$

The onset and bifurcation points will be solved in terms of R, and the corresponding r values are obtained by (1).

Let a, b and c be the three points in the 3-cycle:

$$b = f(a), c = f(b), a = f(c). \tag{2}$$

We define the cyclic polynomials $X = a + b + c$, $X_k = a^k + b^k + c^k$ (for $k = 2, 3$), $Y = ab + bc + ca$, and $Z = abc$. The equations of X, X_k, Y, and Z are more helpful than those of a, b and c in determining the two points of the 3-cycle.

Onset point. From (2), we have $c - b = f(b) - f(a) = -(b + a)(b - a)$; similar, $a - c = -(c + b)(c - b) = (c + b)(b + a)(b - a)$. Now $(b - a) + (c - b) + (a - c) = 0$ means

$$1 - (b + a) + (c + b)(b + a) = 0.$$

Cycling variables $a \to b$, $b \to c$, $c \to a$ twice, and adding up the three versions yields

$$3 - 2X + X^2 + Y = 0. \tag{3}$$

On the other hand, the sum of (2) gives $X = 3R - X_2 = 3R - X^2 + 2Y$ (for $X^2 = X_2 + 2Y$). Using (3) for Y yields

$$X^2 - X + 2 - R = 0, \tag{4}$$

This equation has a real root only if $2 - R \le 1/4$. Thus, $R = 7/4$ (so $r = 1 + \sqrt{1 + 4R} = 1 + \sqrt{8}$) is the onset point of the only real 3-cycle [1–5].

Bifurcation point. Let us express cyclic polynomials as linear functions of X. Using (3) and (4), we get

$$Y = X - R - 1, \tag{5}$$

and

$$X_2 = X^2 - 2Y = 3R - X, \tag{6}$$

For X_3, we have $a^3 = Ra - ab$ from $b = f(a)$, summing over cyclic versions yields

$$X_3 = RX - Y = (R - 1)X + R + 1. \tag{7}$$

By $X_3 - 3Z = X(X_2 - Y)$ and (4)-(7), we have

$$Z = -RX + R - 1. \tag{8}$$

Now (4) can be rewritten in terms of Z:

$$R^3 - 2R^2 + (1 + Z)R + (1 + Z)^2 = 0. \tag{9}$$

The composite map $f(f(f(x)))$ is marginally stable at the onset (bifurcation) point, so $\frac{d}{da} f(f(f(a))) = f'(c)f'(b)f'(a) = +1(-1)$ [1]. As $f'(x) = -2x$, we get

$$Z = abc = \mp 1/8. \tag{10}$$

At the onset point $Z = -1/8$, (9) becomes

$$(R - 7/4)(R^2 - R/4 + 7/16) = 0,$$

whose real solution $R = 7/4$ agrees with the previous result. At the bifurcation point $Z = 1/8$, we have

$$R^3 - 2R^2 + 9R/8 - 81/64 = 0.$$

The only real solution is $R = \frac{2}{3} + \frac{1}{4}\sqrt[3]{\frac{1915}{54} + \frac{5}{2}\sqrt{201}} + \frac{1}{4}\sqrt[3]{\frac{1915}{54} - \frac{5}{2}\sqrt{201}}$ [3, 4].

Since a, b and c are the distinct roots of

$$x^3 - Xx^2 + Yx - Z = 0.$$

The discriminant Δ is a square:

$$X^2Y^2 - 4Y^3 - 4X^3Z + 18XYZ - 27Z^2 = (4X^2 - 6X + 9)^2,$$

where we have used (5) and (8), and eliminated R by (4). Thus, as long as X is real [which is true after the onset $R \ge 7/4$ by (4)], we have $\Delta > 0$, and the cycle points are real. (Thank Beiye Feng for suggesting this point).

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